QU "Geometry & Hamiltonian Differential Equations" (2018/19 · IIa):

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① Let x, y, z be global (e.g., Cartesian) coordinates on $\mathbb{R}^3$ and let $a(x,y,z) \in C^2(\mathbb{R}^3 \to \mathbb{R})$. By definition, for any $f, g \in C^{2\ldots\infty}(\mathbb{R}^3 \to \mathbb{R})$ put
$$\{f,g\} := \left|\frac{\mathcal{D}(f,g,a)}{\mathcal{D}(x,y,z)}\right| = \begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z \end{vmatrix},$$
i.e. the Jacobian.

Prove that $\{\cdot,\cdot\}$ is a Poisson bracket.

(Hint: Show first that in the LHS of the Jacobi identity, all terms with a second derivative of f, g, h or a cancel out.

- List all essentially different types of such terms, e.g.,
  $f_{xx}\, g_y\, a_z \cdot h_y\, a_z$, $f_{xy}\, g_x\, a_y \cdot h_z\, a_z$, etc. — how many?)

Conclusion: There exist $\gg$ Poisson brackets than the inverses $(dp \wedge dq)^{-1}$ of nondegenerate symplectic structures $dp \wedge dq$ on even-dimensional spaces $\mathbb{R}^{2s}$. E.g., the Poisson bi-vector $\mathcal{P} = da / dx \wedge dy \wedge dz$ in the above problem lives on an odd-dimensional space.

Bonus: Its generalisation $da / d\mathrm{vol}(\underline{x})$ is also Poisson, here $d\mathrm{vol}(\underline{x}) = \varrho(\underline{x})\, dx \wedge dy \wedge dz$ with some ($\forall$) nonvanishing density ϱ.

NB. The task of classification of "all" Poisson brackets on $\mathbb{R}^3$ is an open problem.

NB. The above example of $\{\cdot,\cdot\}$ with the Jacobian $\left|\frac{\mathcal{D}(\cdot,\cdot,a)}{\mathcal{D}(x,y,z)}\right|$ of $n=2$ arguments f, g is the simplest example of n-ary Nambu bracket.

NB. Remember $\{\cdot,\cdot\}_\varrho$; it is an excellent "generic" Poisson bracket to work with identities which must hold for all Poisson brackets: in this case, they must hold identically w.r.t. all the derivatives of arbitrary functions a & ϱ.

② Let x, y, z be Cartesian coordinates in $\mathbb{R}^3$ and $\vec{A} = -\mathcal{H} y \cdot \vec{e}_x$ be the "vector potential" of magnetic field $\vec{\mathcal{H}} = \vec{\nabla} \times \vec{A} = \operatorname{rot} \vec{A}$.

- Show that this is a constant homogeneous magnetic field (here $\mathcal{H} = \text{const} \in \mathbb{R}$) so that the motion of an electron $(-e_0)$ must be uniform rectilinear along the field and uniform circular in the cross plane.
- $$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x} - \frac{e_0}{c}\mathcal{H} y\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = 0.$$

  Find the complete integral of the Hamilton-Jacobi equation.
- Using the Jacobi theorem, find the law of motion (see above).

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③ In Kepler's problem of orbital motion, consider the shifted potential $U = -\frac{\alpha}{r} + \delta U(r)$, where (a) $\delta U(r) = \beta / r^2$, $\beta \to 0$.
(b) $\delta U(r) = \gamma / r^3$, $\gamma \to 0$.

The trajectories are no longer closed ellipses; in one period (in $r \Big|_{r_{min}}^{r_{max}}$), the perihelion is shifted by $\Delta\varphi$. Find $\Delta\varphi$ (assume $\beta \to 0$; $\gamma \to 0$).

Bonus*: Find the exact solution $r = r(\varphi)$ for $U = -\frac{\alpha}{r} + \beta / r^2$ by, actually, calculating the same integral as we do at $\beta \equiv 0$.

.* Show that $\Delta\varphi = \dfrac{2\pi}{\sqrt{1 + 2m\beta / M^2}} - 2\pi$.

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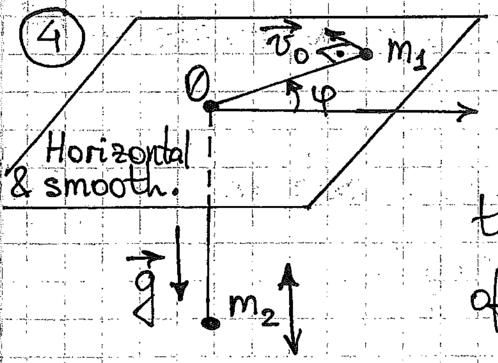
④ 

$v_0 = r_0 \omega_0$; $\dot{r}(t=0) = 0$. $\left\{\begin{array}{l}\text{Length of the}\\ \text{rope} = \ell.\end{array}\right.$

Prove that the domain of motion of m_2 is the annulus bounded by two concentric circles of radii r_0 (see above) and $r_1 = ?$, $r_1 > 0$.

(Hint: • Write down the balance of energy and angular momentum.
• Guess (see above) one positive root of the cubic equation.
• Use the Viet theorem; solve the quadratic eqn, take the only remaining ⊕ve root.)

Ⓐ Prove!

(5) A satellite was orbiting the Earth along circular orbit, which then changed such that the satellite began to oscillate (around/near) the old trajectory. Find the dependence of frequency of these oscillations on their amplitude: $\omega = \omega(a_0)$.

(a) Show that the radial motion satisfies the equation $m\ddot{r} = -\partial U(r)/\partial r$, where $U(r) = \frac{M^2}{2mr^2} - \frac{mgR_0^2}{r}$, here g is the free fall acceleration at the ocean level R_0 (from the Earth centre).

(b) Denote by R the radius of unperturbed circular orbit. Show that $M = m\omega_0 R^2$ and $U(r) = m\omega_0^2 R^2 \cdot \left(\frac{R^2}{2r^2} - \frac{R}{r}\right)$.

(c) Put $r = R + x$; Taylor expand $-\partial U(r)/\partial r$ near $r = R$ up to $\bar{o}(x^3)$.

Prove $$-\frac{\partial U}{\partial r}\Big|_{r=R+x} = -m\omega_0^2 x + \frac{3m\omega_0^2}{R}x^2 - \frac{6m\omega_0^2}{R^2}x^3 + \bar{o}(x^3).$$

(d) $$\ddot{x} + \omega_0^2 x \overset{A}{=} \varepsilon Q(x) = \frac{3\omega_0^2}{R}x^2 - \frac{6\omega_0^2}{R^2}x^3.$$

Apply the Kryloff - Bogolyubov method:

search $x \overset{B}{=} a\cos\psi + \left[\text{discard } \varepsilon\xi_1(a,\psi) \text{ at } t \gg \frac{2\pi}{\omega_0}\right] + \bar{o}(\varepsilon)$,

$$\begin{cases} \dot{a} = \varepsilon f_1(a) + \bar{o}(\varepsilon) \\ \dot{\psi} \overset{C}{=} \omega = \omega_0 + \varepsilon\omega_1(a) + \bar{o}(\varepsilon) \end{cases}$$

- Expand $\varepsilon Q(a\cos\psi)$ into Fourier polynomial.
- Find $a(t, a_0)$.

(e) Show that $$\omega = \omega_0 \cdot \left(1 + \frac{9}{4}\left(\frac{a_0}{R}\right)^2\right),$$

where a_0 is the amplitude of oscillations.

Good luck!